On the Albanese fibration for singular varieties with $-K_X$ nef.

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Setup: $/\mathbb{C}$, $X$ normal proj.

The Albanese: There exists an AV $\mathrm{Alb}_X$ s.t., if we fix $x \in X$

- $\exists!\ \mathrm{alb}_X : X \longrightarrow \mathrm{Alb}_X$ s.t. $\mathrm{alb}_X(x) = 0$, $\mathrm{alb}_X(X)$ generates $\mathrm{Alb}_X$
- $\forall f : X \longrightarrow A$ AV, s.t. $f(x) = 0_A$, $\exists!$ $\varphi$ morphism of AV

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{alb}_X"'] & & A \\ & \mathrm{Alb}_X \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Rmk.:

- $\mathrm{Alb}_X \cong (\mathrm{Pic}^0_X)^\vee$
- $x$ does not matter up to translation
- if $X$ smooth: $X \longrightarrow H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z})$

$$\tilde{x} \longmapsto \left( \omega \longmapsto \int_x^{\tilde{x}} \omega \right)$$

- $C$ smooth curve, $\mathrm{Alb}_C = \mathrm{Jac}(C)$
- $\mathrm{alb}_X$ contracts rat'l curves

Recall $D$ is <u>nef</u> if $D \cdot C \geq 0$ for any $C \subseteq X$ curve

(e.g., $D$ ample $\Rightarrow$ $D$ nef)

<u>Today</u>: $-K_X$ nef

<u>Theorem</u> (Cao): $X$ smooth $-K_X$ nef. Then,

$alb_X : X \to Alb_X$ is surjective, it is a locally analytically trivial fibration (e.g.: $F \times \Delta^n \longrightarrow \Delta^n$)

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<u>Singularities</u>: A pair $(X, D)$ is the datum of $X$, $D \geq 0$ $\mathbb{Q}$-divisor s.t. $K_X + D$ is $\mathbb{Q}$-Cartier

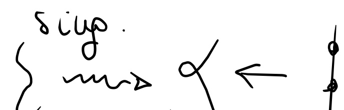
Given $(X, D)$, consider $\pi : Y \to X$ birational s.t.

- $Y$ smooth
- $\mathrm{Supp}(\pi_*^{-1} D) \cup \mathrm{Ex}(\pi)$ snc

Write $\pi^*(K_X + D) = K_Y + D_Y$

We say $(X, D)$ is klt (lc) if $D_Y < 1$ $(D_Y \leq 1)$

<u>Examples</u>:
- cone over rat'l normal curve is klt
- cone over $E \subseteq \mathbb{P}^2$ is lc, not klt
- cone over $C$, $g(C) \geq 2$ not lc
- smooth // reductive, klt
- $X$ smooth, $\mathrm{Supp}\, D$ snc, $(X, D)$ klt $0 \leq D < 1$
  lc $0 \leq D \leq 1$

<u>Motivation</u>:
- $U$ smooth not proper, $\exists\, U \subset X$ s.t. $X$ smooth proper, $X \setminus U$ snc div
  Study $U$ via $(X, X \setminus U)$
- understand quotients
- work w/ res of sing.
- moduli spaces: 
- $P$ prime in $X$, $(K_X + P)|_P = K_P + D$

Theorem (Cadogni–Patakfalvi–Zdanowicz):
$(X,D)$ is klt, $-(K_X+D)$ nef, then
$alb_X : X \to Alb_X$ is locally isotrivial for $(X,\Delta)$
(i.e., locally $(F,D_F)\times\Delta^n \longrightarrow \Delta^n$)

Theorem (Cheng–Zhang): $(X,D)$ lc, $-(K_X+D)$ nef, then
$alb_X$ is surj. with $alb_{X*}\mathcal{O}_X = \mathcal{O}_{Alb_X}$

Theorem (B–PT): $(X,D)$ lc, $-(K_X+D)$ nef, then
$alb_X$ is a locally stable family
- flat
- all fibers are slc

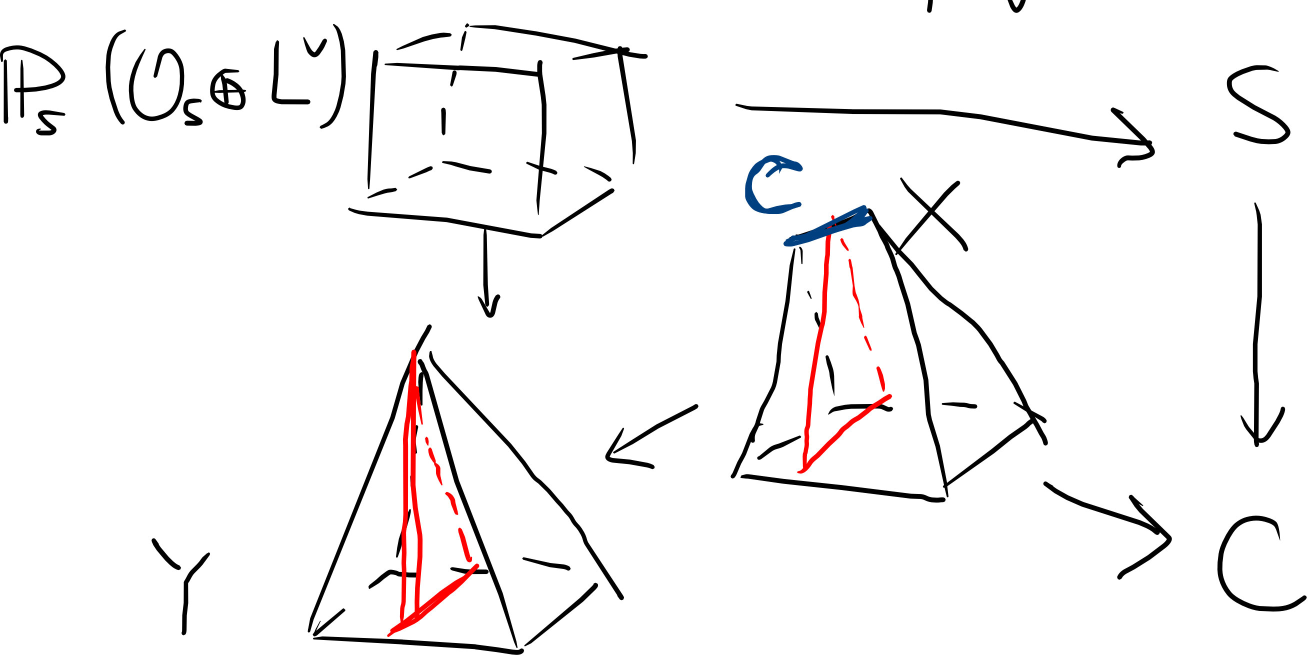
**Example 1:** Fix $C$, $g(C) \geq 2$. Say $g(C) = 2$

$S = C \times C$, take the cone over $S$ with polariz.

$L = p_1^* \omega_C \otimes p_2^* \omega_C^{\otimes 2}$, i.e.,

$$\operatorname{Spec} \bigoplus_{m \geq 0} H^0(S, L^{\otimes m})$$

Let $Y$ be its projective closure. $K_Y$ not $\mathbb{Q}$-Cartier




$-K_X$ ample

$X$ not lc

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d, "alb_C"'] & X \arrow[d, "alb_X"] \\
Alb_C \arrow[r, "\sim"] & Alb_X
\end{tikzcd}$$

Example 2: There exists a log canonical 4-fold $X$ s.t.

- $K_X \sim 0$
- $alb_X : X \longrightarrow E$
- every fiber of $alb_X$ is birat'l to finitely many fibers
- $\pi_1(X_{reg}) \cong \pi_1(E)$ via $alb_X$. In particular, every quasi étale cover of $X$ comes by base change from an étale cover of $E$. So, at the level of (orbifold) universal covers, $\tilde{X} \longrightarrow \mathbb{C}$, two general fibers are not birat'l

- rather than $X$, construct some $(X', D')$ lc (plt)
- we will produce a pencil over $\mathbb{P}^1$, then base change to $E$

Idea: smooth quartic 3-folds are birationally rigid.
So, iso $\Longleftrightarrow$ birat'l

Fact: two irred hypersurfaces in $\mathbb{P}^N$, $N \geq 4$
are isomorphic $\Longleftrightarrow$ $PGL(N+1)$ equivalent

sm 4tic 3 folds birat'l $\Longleftrightarrow$ $PGL(5)$ equivalent

$$\dim PGL(5) = 25 - 1 = 24$$

Sketch: Take $f$ general quartic, $q_1, q_2$ general cubics

- cubics in $x_0, \dots, x_4$ have dim $\binom{7}{4} = 35 > 24$
- $V(f + x_4 q_i)$ smooth quartic 3-fold $\leftarrow$ not birat'l
- $V(x_4) \cap V(f) = V(x_4) \cap V(f + x_4 q_i)$ smooth K3
- $V(x_4) \cap V(q_1 - q_2)$ smooth cubic surface

$$\mathcal{X} = V(y_0 (f + x_4 q_1) + y_1 (f + x_4 q_2)) \subseteq \mathbb{P}^4_{\underline{x}} \times \mathbb{P}^1_{\underline{y}}$$
$$\downarrow$$
$$\mathbb{P}^1_{\underline{y}}.$$

$$D = V(x_4) \cap \mathcal{X}, \quad (\mathcal{X}, D) \xrightarrow{\pi} \mathbb{P}^1$$

Need: $K_{\mathcal{X}} + D \sim \pi^* \omega_{\mathbb{P}^1}$

Have: $K_{\mathcal{X}} + D \sim \pi^* \mathcal{O}_{\mathbb{P}^1}(-1)$, over $[1 : -1]$ bad fiber

Replace birationally $\mathcal{X}$ to make $\mathcal{X}_{[1:-1]}$ irreducible,
remove $D^v$